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On Dominant Integrability

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The property of dominant integrability was introduced by C. F. Osgood and O. Shisha [6]. One of the motivations in introducing this concept was the problem: under what conditions can quadrature formulas effective for Riemann integrable functions on [0, 1] be used for the numerical evaluation of improper Riemann integrals $\int_{0+}^{1} f(x)\,dx$? It turns out that, for a function f on (0, 1], its dominant integrability is a necessary and sufficient condition for $\int_{0+}^{1} f(x)\,dx$ to converge and to equal $\lim_{n \to \infty} \Phi_n^*(f)$ for every sequence $(\Phi_n^*)_{n=1}^{\infty}$ of quadrature formulas of a very general type. (For details see [6, 7].)

One of the definitions of dominant integrability is the following ([6], second line of Corollary 2, Definition 4, and Theorem 3): *Let f be a complex function on* (0, 1]. *Dominant integrability of f means Riemann integrability of f on each closed subinterval of* (0, 1] *and the existence of a real, nonnegative function g, monotone nonincreasing on* (0, 1], *with* $\int_{0+}^{1} g(x)\,dx < \infty$, *satisfying throughout* (0, 1], $|f(x)| \leq g(x)$.

Dominant integrability turns out to be a very simple instance of generalized Riemann integrability, an elementary property equivalent to Perron and restricted Denjoy integrability but more general than Lebesgue integrability. For details see [1, 3]. The simplicity and great power of the concept of generalized Riemann integrability should make it the standard concept of integrability.

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Our main purpose here is to state and prove

THEOREM 1. *Let f be a complex function on $(0, 1]$ and let $0<\delta<1$. A necessary and sufficient condition for f to be dominantly integrable is that it be Riemann integrable on each $[a, 1]$, $0<a<1$, and that*

$$\sum_{j=1}^{\infty} w(f, \delta^j, \delta^{j-1})\,\delta^j < \infty,$$

where $w(f, \delta^j, \delta^{j-1})$ is the oscillation of f on $[\delta^j, \delta^{j-1}]$, namely,

$$\sup\{|f(t_1)-f(t_2)| : \delta^j \leqslant t_1 \leqslant t_2 \leqslant \delta^{j-1}\}.$$

Proof. Necessity: By the definition of dominant integrability in Section 1, f is Riemann integrable (and, hence, bounded) on each $[a, 1]$, $0<a<1$. For every $t \in (0, 1]$, let

$$\hat{f}(t) = \sup\{|f(x)| : t \leqslant x \leqslant 1\}.$$

For $j = 1, 2, \ldots$, if $\delta^j \leqslant t_1 \leqslant t_2 \leqslant \delta^{j-1}$, then

$$|f(t_1)-f(t_2)| \leqslant |f(t_1)| + |f(t_2)| \leqslant 2\hat{f}(\delta^j)$$

and, hence,

$$w(f, \delta^j, \delta^{j-1}) \leqslant 2\hat{f}(\delta^j).$$

Thus, for $n = 1, 2, \ldots$,

$$\begin{aligned}\sum_{j=1}^{n} w(f, \delta^j, \delta^{j-1})\,\delta^j &\leqslant 2\sum_{j=1}^{n} \hat{f}(\delta^j)\,\delta^j = 2\sum_{j=2}^{n+1} \hat{f}(\delta^{j-1})\,\delta^{j-1}\\ &= 2(1-\delta)^{-1}\sum_{j=2}^{n+1} \hat{f}(\delta^{j-1})(\delta^{j-1}-\delta^j)\\ &\leqslant 2(1-\delta)^{-1}\int_{\delta^{n+1}}^{\delta} \hat{f}(x)\,dx\\ &\leqslant 2(1-\delta)^{-1}\int_{0+}^{\delta} \hat{f}(x)\,dx < \infty\end{aligned}$$

(see [6], Definition 1, Corollary 2 and the first sentence of its proof). Hence the desired conclusion.

Sufficiency: For $n=1, 2, \ldots$ and $x \in (0, 1]$, let

$$g_n(x) = \begin{cases} \sum_{j=1}^{n} w(f, \delta^j, \delta^{j-1}) & \text{if} \quad \delta^n < x \leqslant \delta^{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

For every $x \in (0, 1]$, say,

$$\delta^n < x \leqslant \delta^{n-1}, \qquad n \geqslant 1 \text{ an integer,} \tag{1}$$

set

$$g(x) = |f(1)| + g_n(x) = |f(1)| + \sum_{k=1}^{\infty} g_k(x)$$

so that g is monotone nonincreasing on $(0, 1]$. Then

$$\begin{aligned} \sum_{j=1}^{\infty} w(f, \delta^j, \delta^{j-1})\, \delta^{j-1} &= (1-\delta) \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} w(f, \delta^j, \delta^{j-1})\, \delta^{k-1} \\ &= (1-\delta) \sum_{k=1}^{\infty} \sum_{j=1}^{k} w(f, \delta^j, \delta^{j-1})\, \delta^{k-1} \\ &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} w(f, \delta^j, \delta^{j-1}) \right) (\delta^{k-1} - \delta^k) \\ &= \sum_{k=1}^{\infty} \int_0^1 g_k(x)\, dx = \int_{0+}^1 g(x)\, dx - |f(1)|. \end{aligned}$$

If $x \in (0, 1]$, say, (1), then

$$\begin{aligned} |f(x) - f(1)| &= \left| f(x) - f(\delta^{n-1}) + \sum_{k=1}^{n-1} f(\delta^k) - f(\delta^{k-1}) \right| \\ &\leqslant |f(x) - f(\delta^{n-1})| + \sum_{k=1}^{n-1} |f(\delta^k) - f(\delta^{k-1})| \\ &\leqslant w(f, \delta^n, \delta^{n-1}) + \sum_{k=1}^{n-1} w(f, \delta^k, \delta^{k-1}) \\ &= \sum_{k=1}^{n} w(f, \delta^k, \delta^{k-1}) = g_n(x) \end{aligned}$$

$\left(\sum_{k=1}^{n-1} = 0 \text{ if } n = 1 \right)$; so

$$|f(x)| \leqslant |f(1)| + g_n(x) = g(x).$$

Hence f is dominantly integrable.

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The concept of dominant integrability has been extended in [4] to complex functions on $I=(0,1]\times(0,1]$. Using [4], one can imitate the proof of Theorem 1 and prove

THEOREM 2. *Let f be a complex function on I and let $0<\delta<1$. A necessary and sufficient condition for f to be dominantly integrable on I is that it be Riemann integrable on each $[a,1]\times[b,1]$, $0<a<1$, $0<b<1$, and that*

$$\sum_{j,k=1}^{\infty} w(f,[\delta^j,\delta^{j-1}]\times[\delta^k,\delta^{k-1}])\,\delta^{j+k}<\infty,$$

where the coefficient of δ^{j+k} is the oscillation of f on

$$R_{j,k}=[\delta^j,\delta^{j-1}]\times[\delta^k,\delta^{k-1}],$$

namely, $\sup\{|f(P_1)-f(P_2)|: P_1, P_2\in R_{j,k}\}$.

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